

# Olympus Protocol: Consensus and Security

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## Abstract

In this paper, we build up the mathematical foundations of Olympus Protocol (OSP), which advanced the directed acyclic graph (DAG) for storing transactions. Olympus has superior performance such as high throughput and almost-zero transaction fees. The nature of its original architecture enables the flexibility and capability of chain interoperability with all other layer-1 blockchains, which is the foundation for building a Web3 data infrastructure. We provide thorough and rigorous analysis on our consensus mechanism which depends on non-anonymous reputable entities, called committees. Our scheme allows committees to be replaced to achieve higher level of decentralization. The security of Olympus Protocol against malicious behaviors is guaranteed.

## 1 Introduction

The concept of blockchain as an independent technology began to surge in 2015. Prior to this, it was known as the data structure of Bitcoin. In Nakamoto’s white paper [1], the two words “block” and “chain” appear together, but it only refers to “a series of blocks.” With the popularity of Bitcoin, the technology and concepts in Bitcoin is often classified as Blockchain 1.0. With Ethereum [2] running as a platform for distributed applications, people began to classify Ethereum as Blockchain 2.0. Now the market is vying for the fundamental structure for a new paradigm of Internet infrastructure, interoperability and scalability, i.e., Blockchain 3.0. Many people think that directed acyclic graph (DAG) structure is one of the best candidates.

In traditional blockchain technology represented by Bitcoin and Ethereum, blocks and transactions are two separate concepts. A transaction is confirmed by the miners and packed into a block, and the throughput in terms

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\*Oort was formerly called “Computecoin Networks”. Please visit our official website to stay up-to-date on our progress, and to view the latest version of this technical paper.

of transactions per second (TPS) is limited by the block size and the block generation speed. In addition, miners in the blockchain system have the right to decide the content of the block. The profit-seeking behavior of the miners can easily lead to excessive concentration of power or voting rights, thus losing the decentralization characteristics. DAG-based distributed ledger technology (DLT) was created to solve these problems. Compared to traditional blockchain technology, DAG-based DLT has the following advantages: 1) Strong scalability (high TPS); 2) Fast transaction speed; 3) (Almost) no transaction fee and friendly to small payments; 3) No requirement for special miners to participate.

The idea of using DAGs in the cryptocurrency space has been around for a while. DAGLabs has proposed a series of consensus protocols, such as Inclusive [3], SPECTRE [4] and PHANTOM [5]. The general idea behind them is to utilize a DAG of blocks. Also the miners in the system still compete for transaction fees, and new tokens may be created by these miners. Instead, some cryptocurrencies depend on a DAG of individual transactions other than blocks. IOTA [6] and Byteball<sup>1</sup> [7] are among the oldest and most representative projects. They both have the same advantages using a DAG structure, but have quite different design details in order to cater to different audiences. IOTA assigns a certain weight to each transaction, and the transaction is generated through the proof of work (PoW) mechanism. Instead of utilizing PoW, Byteball prevents junk transactions by charging a small fee, and introduces votes from committees to determine valid transactions.

Similar to IOTA and Byteball, transactions in OSP are stored and organized in a DAG structure. However, we impose some additional rules, which results in a special DAG called OSP directed acyclic graph (OSP-DAG). Consensus in our OSP-DAG is achieved through committees, which are non-anonymous reputable entities. It is a Byzantine Fault Tolerant (BFT) consensus protocol which can tolerate malicious behaviors. Since the FLP impossibility result [8] has demonstrated the impossibility of distributed consensus in an asynchronous environment, we assume one of the two forms of partial synchrony defined in [9]. That is, the upper bound on the time required for a message to be delivered is fixed but not known a priori. The main advantage of our consensus algorithm, compared with the state-of-the-art BFT protocols such as PBFT [10] and Tendermint [11], is the exclusion of additional messages for voting purpose. It significantly reduces the communication overhead, which in turn alleviates the scaling

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<sup>1</sup>Byteball project has been renamed as Obyte.

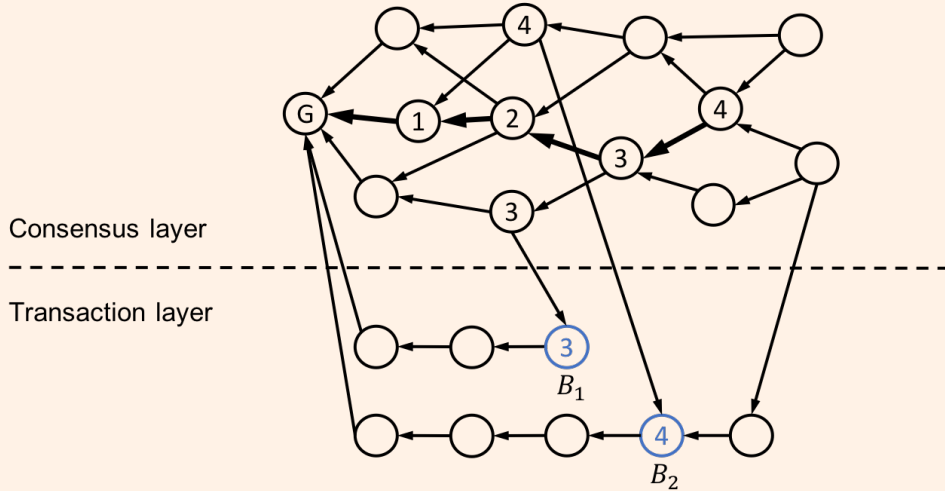


Figure 1: Example of consensus in OSP-DAG structure

issues to achieve higher TPS.

The remainder of the paper is organized as follows. The OSP-DAG structure is presented in Section 2. The proposed consensus algorithm is described in Section 3. Section 4 rigorously proves the correctness of our consensus protocol, including both safety and liveness properties.

## 2 OSP-DAG

In OSP, each block represents one transaction, which contains references to previous blocks (called parents) through their hashes. Blocks and their parent-child links are the vertices and edges of the DAG, respectively. As depicted in Fig. 1, our OSP-DAG structure has two layers, namely the consensus layer and the transaction layer.

All blocks in the consensus layer are composed by some non-anonymous reputable people or companies, called committees, who might have a long established reputation, or great benefits in keeping the network healthy. Each block in the consensus layer can reference multiple blocks from both the consensus layer and the transaction layer. committees are expected to post transactions frequently and behave honestly. However, it is unreasonable to totally trust any single committee. Our proposed scheme allows committees to be replaced without jeopardizing the consensus and security in the network. Details on how to change committees will be elaborated

in Section 3. Transactions in the consensus layer is for the sole purpose of achieving consensus in the network, while real transactions happen in the transaction layer. In the transaction layer, each account has its own chain of blocks, which records the transaction history of this account. In addition, each block in the transaction layer is referenced by blocks in the consensus layer.

The consensus in the computecoin network is achieved via total ordering of all blocks. Each node starts by finding out the “stable” main chain within the consensus layer of its local DAG. The rigorous definition of stable main chain will be described later in Section 3.1. Each node then numbers all blocks included by blocks on the stable main chain as follows. It first defines indices for blocks that lie directly on the stable main chain. The genesis block has index 0, the next block on the stable main chain that is a child of the genesis block has index 1, and so on. By traveling forward along the stable main chain, it assigns indices to blocks that lie on the stable main chain. For any block that does not lie on the stable main chain, its index is assigned by the index of the block on the stable main chain that first references it directly or indirectly. Now each node can determine the order for any two blocks  $B_1$  and  $B_2$  with assigned indices using the following rule  $\mathcal{O}$ :  $B_1$  precedes  $B_2$  if and only if

- a)  $B_1$  has lower index than  $B_2$ ; or
- b)  $B_1$  and  $B_2$  have the same indices, but  $B_1$  is referenced by  $B_2$  directly or indirectly; or
- c)  $B_1$  and  $B_2$  have the same indices, and there is no reference relationship between  $B_1$  and  $B_2$ , but  $B_1$  has lower hash than  $B_2$ .

As a concrete example shown in Fig. 1, a node is trying to decide the order of two blocks  $B_1$  and  $B_2$  marked in blue. The stable main chain it finds out is marked in bold arrows. And the numbers inside each block are indices assigned according to the stable main chain. Now block  $B_1$  has index 3 and block  $B_2$  has index 4. Therefore, the node will determine that  $B_1$  precedes  $B_2$  since  $B_1$  has lower index than  $B_2$ .

### 3 Consensus in OSP

In this section, we will focus on the consensus layer of our OSP-DAG structure, and explain in detail how a node finds out the stable main chain of its local graph. The remainder of this section is organized as follows. The

key terms which will be used intensively throughout the paper are described in Section 3.1. In Section 3.2, we list the key assumptions we rely on in order to guarantee that the computecoin network is secure. Based on the definitions and assumptions, Section 3.3 presents the consensus algorithm which is implemented in the computecoin mainnet.

### 3.1 Definitions

At any time, each node in the network would observe slightly different graph due to network delay. Let  $G_n(t)$  denote the graph node  $n$  has observed at time  $t$ . In this section, we drop  $n$  and  $t$  and use  $G$  to represent a general DAG which satisfies that if a block  $B$  is in  $G$ , all  $B$ 's parents are also in  $G$ . In the following, we describe some key terms which will be used intensively in the subsequent sections.

- D1 Graph inclusion relation: We use  $G \subseteq G^*$  to represent that  $G^*$  contains all blocks in  $G$ , and  $G^*$  satisfies the condition that if a block  $B$  is in  $G^*$ , all  $B$ 's parents are also in  $G^*$ .
- D2 Block inclusion relation: We say a block  $B_1$  includes another block  $B_0$  if  $B_1 = B_0$  or  $B_1$  references  $B_0$  directly or indirectly.
- D3 Block comparison: Suppose each block in  $G$  has its epoch, level and hash, where the definitions of epoch and level will be discussed in D6 and D7, respectively. For any pair of blocks  $B_0$  and  $B_1$ , we call  $B_1$  is better than  $B_0$  if and only if  $B_1$  has larger epoch, or larger level if  $B_0$  and  $B_1$  have the same epoch, or larger hash in the case that  $B_0$  and  $B_1$  have the same epoch and the same level. We denote this comparison rule as  $\mathcal{R}$ .
- D4 Best Parent: The best parent of a block is one of its parents, which is the best under block comparison rule  $\mathcal{R}$ . The best parent of a block  $B$  is denoted by  $\text{bp}(B)$ .
- D5 Block height: The height of a block  $B$ , denoted by  $\text{h}(B)$ , refers to the length of the path from  $B$  to the genesis block through best parent links. Note that the height of the genesis block is 0.
- D6 Epoch: The system moves through a succession of configurations called epochs. In each epoch, there is a different set of committees, denoted by  $\mathcal{W}_i$ . Let  $N_i$  denote the number of committees in  $\mathcal{W}_i$  and  $K_i = \lfloor \frac{2}{3}N_i \rfloor + 1$ . We represent the set of all nonnegative integers as a union

of disjoint consecutive integer sequences, i.e.,  $\mathbb{N} \cup \{0\} = \bigcup_{i=1}^{\infty} \mathcal{I}_i$ , where  $\mathcal{I}_i$  is a consecutive integer sequence ranging from  $a_i$  to  $b_i$ . Here, all the numbers in  $\mathcal{I}_j$  is larger than those in  $\mathcal{I}_i$  for any  $j > i$ , i.e.,  $a_j > b_i$ . The epoch a block  $B$  belongs to is determined by which interval the height of the last stable block (defined later in D10) of  $B$ 's best parent falls in. Specifically, if the height of the last stable block of  $\text{bp}(B)$  is in  $\mathcal{W}_i$ , the epoch of block  $B$ , denoted by  $\text{ep}(B)$ , is  $i$ .

D7 Block level: The level of a block  $B$ , denoted by  $\text{lv}(B)$ , is defined as follows:

$$\text{lv}(B) = \begin{cases} 0, & \text{if } B \text{ is the genesis block,} \\ 1, & \text{if } \text{ep}(B) > \text{ep}(\text{bp}(B)), \\ \text{lv}(\text{bp}(B)) + 1, & \text{if } \text{ep}(B) = \text{ep}(\text{bp}(B)). \end{cases} \quad (1)$$

D8 Main chain: The main chain of graph  $G$  is defined as the path starting from the best tip block in  $G$  under block comparison rule  $\mathcal{R}$  to the genesis block through best parent links. Here, tip blocks refer to blocks without any child.

D9 Stable block: A block on the main chain of  $G$  is called a stable block of  $G$  if it is guaranteed to be contained in the main chain of any graph  $G^*$  that includes  $G$ , i.e.,  $G \subseteq G^*$ .

D10 Last stable block: The last stable block of the genesis block is itself. Now for a block  $B_1$ , given that the last stable block of its best parent is defined, the last stable block of  $B_1$  is determined by the following procedure. For any two blocks  $B$  and  $B^*$ , we use  $B^* \rightarrow B$  to denote that  $B^*$  includes  $B$  through parent links and all blocks in the path (including both  $B^*$  and  $B$ ) must be in the same epoch. Similarly, we use  $B^* \xrightarrow{b} B$  to denote that  $B^*$  includes  $B$  through best parent links and all blocks in the path need not be in the same epoch. The degenerated case of  $B = B^*$  is regarded true, i.e.,  $B^* \rightarrow B$  and  $B^* \xrightarrow{b} B$ . For any block  $B_0$  such that  $B_1 \xrightarrow{b} B_0$ , let  $C(B_0, B_1)$  denote the set of blocks from  $B_1$  to  $B_0$  through best parent links, which includes  $B_1$  but not  $B_0$ . Assume  $\text{ep}(B_1) = i$ . Start with  $B_0 = \text{lsb}(\text{bp}(B_1))$ , and check whether the following condition holds

$$\text{lv}(B_1) > \max_{B \in S(B_0, B_1)} \text{lv}(B) + 2(K_i - 1), \quad (2)$$

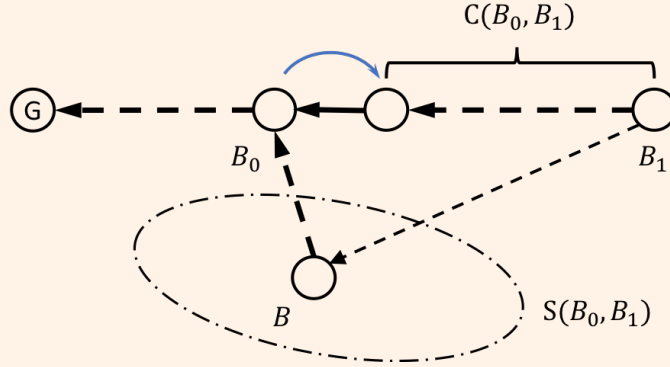


Figure 2: One step in finding out the last stable block of  $B_1$ . Solid and dashed lines represent parent-child links and ancestor-descendant links, respectively. Bold and regular lines represent  $\xrightarrow{b}$  and  $\rightarrow$  relations, respectively.

where  $S(B_0, B_1) = \{B \mid B \xrightarrow{b} B_0, B_1 \rightarrow B, C(B_0, B) \cap C(B_0, B_1) = \emptyset\}$ . If  $S(B_0, B_1) = \emptyset$ , the maximal value over  $S(B_0, B_1)$  in (2) is set to be 0. If the condition (2) holds, update  $B_0$  to be its child on  $C(B_0, B_1)$  and go back to check the condition (2) again, and so on. We repeatedly advance  $B_0$  till  $h(B_0) \in \mathcal{I}_{i+1}$  or  $B_1$  does not satisfy the condition (2) with respect to  $B_0$ . The block  $B_0$  we stop at is the last stable block of  $B_1$ , denoted by  $\text{lsb}(B_1)$ . One advancement of  $B_0$  described above is depicted as the blue arrow in Fig. 2.

- D11 **Stable main chain:** From the last stable blocks of all blocks in  $G$ , we pick the one with the largest height, denoted by  $\text{SB}(G)$ . The stable main chain of  $G$ , denoted by  $\text{SC}(G)$ , is then defined as the chain of blocks starting from  $\text{SB}(G)$  to the genesis block through best parent links. Note that the stable main chain of  $G$  is part of the main chain that will not change as  $G$  expands.
- D12 **Main chain index (MCI):** The MCI for any block that lies directly on the stable main chain is equal to its height. For any block that does not lie on the main chain, its MCI is assigned by the MCI of the block on the stable main chain that first includes it. The MCI of a block  $B$  is denoted by  $\text{mci}(B)$ .

Many definitions above depend on each other. However, they can be incrementally built up as the DAG grows. To start with, the genesis block belongs to epoch 0, has level 0 and its last stable block is itself. For a new

block  $B$  added to the graph, assume that all terms for its parents are already well defined. We first find out its best parent  $\text{bp}(B)$  via block comparison rule  $\mathcal{R}$ . Next, we find out its epoch  $\text{ep}(B)$  by checking the height of the last stable block of  $\text{bp}(B)$ .  $B$ 's level  $\text{lv}(B)$  can then be determined by (1). And the last step is to find out the last stable block of  $B$ , i.e.,  $\text{lsb}(B)$  by the procedure described in D10. After that, we will know whether the stable main chain of the graph has been extended or not.

### 3.2 Assumptions

The key assumptions used in OSP consensus protocol and subsequent technical discussions are as follows:

- A1 Honest committees should generate blocks serially. In other words, each honest committee should reference (directly or indirectly) all its previous blocks in every subsequent block.
- A2 When an honest committee composes a block, he always chooses the best tip block of its local graph under block comparison rule  $\mathcal{R}$  as the best parent of this new block.
- A3 If a block is in epoch  $i$ , the issuer of this block must be in the committee set  $\mathcal{W}_i$ .
- A4 Start from any block in epoch  $i$  and traverse through best parent links, we stop as soon as we encounter  $K_i$  blocks or a block of level 1, whichever comes first. Each block we encountered (including the one we stop at) must be issued by a different committee from the committee set  $\mathcal{W}_i$ .
- A5 In each epoch  $i$ , more than  $2/3$  of the committees in  $\mathcal{W}_i$  are honest. In other words, at least  $K_i$  committees are honest, where  $K_i = \lfloor \frac{2}{3}N_i \rfloor + 1$  is defined in D6.
- A6 Any block will be delivered to all honest committees within some fixed but unknown amount of time. It implies that for honest committees, the graphs they eventually observe would be consistent with each other. That is to say, for any pair of honest committees  $i$  and  $j$ , the graph  $\mathbf{G}_i(t_i)$  node  $i$  observed at time  $t_i$  will also be observed by node  $j$  at some time  $t_j$ , i.e.,  $\mathbf{G}_i(t_i) \subseteq \mathbf{G}_j(t_j)$ .

The assumptions from A1 to A4 are also constraints that need to be satisfied when a committee issues a block. Among those, however, only A3



and A4 are binding. That is to say, other committees can perform certain sanity check on A3 and A4, and reject the block if either of these two conditions is not met. Note that assumption A6 is a form of partial asynchrony [9], which is a middle ground between synchrony and asynchrony.

### 3.3 Consensus Algorithm

Based on the definitions and assumptions above, the consensus algorithm implemented in OSP is summarized in Algorithm 1. The key idea is on how to consistently expand the local graph when receiving a block. For consensus purpose, we only need to deal with blocks issued by committees and update the stable main chain accordingly, since only those blocks can contribute to the consensus of the system.

## 4 Correctness

This section provides the technical proofs to show that the consensus algorithm described in Algorithm 1 is correct. Section 4.1 provides some useful propositions that will be used in the subsequent sections. In Section 4.2, we show that the advance of last stable block defined in D10 guarantees that the last stable block is indeed stable. Section 4.3 and Section ?? are dedicated to prove that our consensus algorithm satisfies safety and liveness properties, respectively. Note that in this section, we still focus on the consensus layer of our OSP-DAG structure.

### 4.1 Propositions

Recall that for any two blocks  $B$  and  $B^*$ ,  $B^* \rightarrow B$  denotes that  $B^*$  includes  $B$  through parent links and all blocks in the path (including both  $B^*$  and  $B$ ) are in the same epoch. Similarly,  $B^* \xrightarrow{b} B$  denotes that  $B^*$  includes  $B$  through best parent links and all blocks in the path are not necessarily in the same epoch. In the following, we prove some useful results which will be used in later analysis.

**Proposition 1.** *For any two blocks  $B_0$  and  $B_1$ , if  $B_0 = \text{bp}(B_1)$ , we have  $\text{lsb}(B_1) \xrightarrow{b} \text{lsb}(B_0)$ , and  $\text{ep}(B_1) = \text{ep}(B_0)$  or  $\text{ep}(B_1) = \text{ep}(B_0) + 1$ .*

*Proof.* It can be directly inferred from how the last stable block is determined as described in D10. To find the last stable block of  $B_1$ , we start with  $B^* = \text{lsb}(B_0)$ , and update  $B^*$  to be its child in  $\mathcal{C}(B^*, B_1)$  in each step

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**Algorithm 1** OSP Consensus Algorithm

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1: *Input:* Local graph  $G = \{G\}$  for some node, where  $G$  is the genesis block  
2: *Initialization:* Set  $\text{ep}(G) = 0, \text{lv}(G) = 0, \text{lsb}(G) = G$ .  
3: *Main iterations:*  
4: **for all** received block  $B_1$  **do**  
5:   **if**  $B_1$  does not pass the sanity checks **then**  
6:     Reject block  $B_1$ .  
7:     Continue  
8:   **end if**  
9:   **if** At least one of  $B_1$ 's parent is not in  $G$  **then**  
10:     Add block  $B_1$  into a buffer for future consideration.  
11:     Continue  
12:   **end if**  
13:   **if**  $B_1$  is not issued by a committee **then**  
14:     Continue  
15:   **end if**  
16:   Determine  $B_1$ 's best parent  $\text{bp}(B_1)$  by block comparison rule  $\mathcal{R}$ .  
17:   Determine  $B_1$ 's epoch  $\text{ep}(B_1)$  by checking which interval the height of  $\text{lsb}(\text{bp}(B_1))$  falls in. Assume the interval is  $\mathcal{I}_i$ , i.e.,  $\text{ep}(B_1) = i$ .  
18:   **if** Assumptions A3 or A4 is not satisfied **then**  
19:     Reject block  $B_1$ .  
20:     Continue  
21:   **end if**  
22:   Add  $B_1$  to  $G$ , and determine  $B_1$ 's level  $\text{lv}(B_1)$  according to (1).  
23:   Set  $B_0 = \text{lsb}(\text{bp}(B_1))$ .  
24:   **while** The condition (2) holds **do**  
25:     Update  $B_0$  to be its child in  $C(B_0, B_1)$ .  
26:   **end while**  
27:   Set  $\text{lsb}(B_1) = B_0$ .  
28:   **if**  $\text{lsb}(B_1)$  has larger height than the tip block of the existing stable main chain **then**  
29:     Update the stable main chain  $\text{SC}(G)$  to end with  $\text{SB}(G) = \text{lsb}(B_1)$ .  
30:   **end if**  
31:   Find out MCIs of all blocks that are included by any block on  $\text{SC}(G)$ .  
32: **end for**  
33: *Output:* Linear ordering of all blocks that are included by any block on  $\text{SC}(G)$  using rule  $\mathcal{O}$ .

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as long as  $B_1$  satisfies the condition (2) with respect to  $B^*$ . It guarantees that in every step, the new  $B^*$  references the old one through the best parent link. Therefore, we have  $\text{lsb}(B_1) \xrightarrow{b} \text{lsb}(B_0)$ . Assume  $\text{ep}(B_0) = i$ , i.e.,  $\text{h}(\text{lsb}(\text{bp}(B_0))) \in \mathcal{I}_i$ . To find the last stable block of  $B_0$ , the block we stop at, i.e.,  $\text{lsb}(B_0)$  must satisfy that  $\text{h}(\text{lsb}(B_0))$  is still in  $\mathcal{I}_i$  or in  $\mathcal{I}_{i+1}$ . It follows that  $\text{ep}(B_1) = i$  or  $i + 1$ , which leads to  $\text{ep}(B_1) = \text{ep}(B_0)$  or  $\text{ep}(B_1) = \text{ep}(B_0) + 1$ .  $\square$

**Proposition 2.** *For any two blocks  $B_0$  and  $B_1$ , if  $B_1$  includes  $B_0$ , we have  $\text{ep}(B_1) \geq \text{ep}(B_0)$ .*

*Proof.* The statement is true for the trivial case  $B_0 = B_1$ . Now we assume that  $B_0 \neq B_1$ . First, we show that if  $B_0$  is a parent of  $B_1$ ,  $\text{ep}(B_1) \geq \text{ep}(B_0)$  holds. Consider the following two cases.

- 1)  $B_0$  is the best parent of  $B_1$ : We have  $\text{lsb}(B_0) \xrightarrow{b} \text{lsb}(\text{bp}(B_0))$  by Proposition 1. It follows that  $\text{h}(\text{lsb}(B_0)) \geq \text{h}(\text{lsb}(\text{bp}(B_0)))$ . Thus, there exists  $i \geq j$  such that  $\text{h}(\text{lsb}(B_0)) \in \mathcal{I}_i$  and  $\text{h}(\text{lsb}(\text{bp}(B_0))) \in \mathcal{I}_j$ . Therefore,  $\text{ep}(B_1) = i \geq j = \text{ep}(B_0)$ .
- 2)  $B_2 \neq B_0$  is the best parent of  $B_1$ : Similarly as in the previous case, we have  $\text{ep}(B_1) \geq \text{ep}(B_2)$ . According to the definition of best parent,  $B_2$  is better than  $B_0$  under block comparison rule  $\mathcal{R}$ . It implies that  $\text{ep}(B_2) \geq \text{ep}(B_0)$ . Therefore, we have  $\text{ep}(B_1) \geq \text{ep}(B_2) \geq \text{ep}(B_0)$ .

For the general case that  $B_1$  does not directly reference  $B_0$ , we can apply the chain rule to show that  $\text{ep}(B_1) \geq \text{ep}(B_0)$ .  $\square$

**Proposition 3.** *For any two blocks  $B_0$  and  $B_1$ , if  $B_1 \rightarrow B_0$ , we have  $\text{lv}(B_1) \geq \text{lv}(B_0)$ .*

*Proof.* The statement is true for the trivial case  $B_0 = B_1$ . Now we assume that  $B_0 \neq B_1$ . First, we show that if  $B_0$  is a parent of  $B_1$ ,  $\text{lv}(B_1) \geq \text{lv}(B_0)$  holds. Consider the following two cases.

- 1)  $B_0$  is the best parent of  $B_1$ : Since  $B_0$  and  $B_1$  are in the same epoch by the definition of  $B_1 \rightarrow B_0$ , we have  $\text{lv}(B_1) = \text{lv}(B_0) + 1 > \text{lv}(B_0)$  by (1).
- 2)  $B_2 \neq B_0$  is the best parent of  $B_1$ : According to the definition of best parent,  $B_2$  is better than  $B_0$  under block comparison rule  $\mathcal{R}$ . It implies that  $\text{ep}(B_2) \geq \text{ep}(B_0)$ . It follows that

$$\text{ep}(B_2) \geq \text{ep}(B_0) \stackrel{(a)}{=} \text{ep}(B_1) \stackrel{(b)}{\geq} \text{ep}(B_2), \quad (3)$$

where (a) is by the definition of  $B_1 \rightarrow B_0$  and (b) is by Proposition 2. Thus, the following condition holds:  $\text{ep}(B_0) = \text{ep}(B_1) = \text{ep}(B_2)$ . Therefore, we have

$$\text{lv}(B_1) \stackrel{(a)}{=} \text{lv}(B_2) + 1 \stackrel{(b)}{\geq} \text{lv}(B_0) + 1 > \text{lv}(B_0), \quad (4)$$

where (a) is by (1) and (b) is due to the fact that  $\text{lv}(B_2) \geq \text{lv}(B_0)$  since  $B_2$  is better than  $B_0$  under  $\mathcal{R}$  but  $\text{ep}(B_0) = \text{ep}(B_2)$ .

For the general case that  $B_1$  does not directly reference  $B_0$ , we can apply the chain rule to show that  $\text{lv}(B_1) \geq \text{lv}(B_0)$ .  $\square$

The following is a direct corollary of Proposition 2 and Proposition 3.

**Corollary 1.** *For any two blocks  $B_0$  and  $B_1$ , if  $B_1$  includes  $B_0$  and  $\text{ep}(B_1) = \text{ep}(B_0)$ , we have  $B_1 \rightarrow B_0$  and  $\text{lv}(B_1) \geq \text{lv}(B_0)$ .*

## 4.2 Advance of Last Stable Block

Let  $\mathbb{G}^B$  denote the induced graph from a block  $B$  in  $\mathbb{G}$  which consists of all blocks that  $B$  includes. In this section, we will analyze the procedure to determine the last stable block of  $B$ , i.e.,  $\text{lsb}(B)$ . Our main goal is to show that  $\text{lsb}(B)$  is a stable block of graph  $\mathbb{G}^B$ . Recall that from Assumption A4, if we start from block  $B$  in epoch  $i$ , traverse through best parents links, and stop as soon as  $K_i$  blocks or a block of level 1 has been visited, all blocks encountered must be issued by different committees from the committee set  $\mathcal{W}_i$ . Let  $\mathbb{T}(B)$  and  $\mathbb{W}(B)$  denote the set of blocks encountered and the set of committees who issue these blocks, respectively. Note that all blocks in set  $\mathbb{T}(B)$  are in the same epoch as  $B$ . In the following, we first prove three lemmas which are crucial for the proof of our claim.

**Lemma 1.** *If  $B_1 \xrightarrow{b} B_0$ , all blocks in  $\mathbb{C}(B_0, B_1)$  are in epoch  $i$  and none of them is issued by an honest committee from a set  $\mathcal{W} \subseteq \mathcal{W}_i$  which consists of  $K_i$  committees, then  $\mathbb{C}(B_0, B_1)$  contains at most  $K_i - 1$  blocks, i.e.,  $|\mathbb{C}(B_0, B_1)| \leq K_i - 1$ .*

*Proof.* Since all blocks in  $\mathbb{C}(B_0, B_1)$  are issued by committees from set  $\mathcal{W}_i$  and none of them is issued by an honest committee from  $\mathcal{W}$ , they can only be issued by  $N_i - K_i$  committees outside  $\mathcal{W}$  and malicious committees inside  $\mathcal{W}$ , which is at most  $N_i - K_i$  by Assumption A5. Thus, due to  $K_i > \frac{2}{3}N_i$

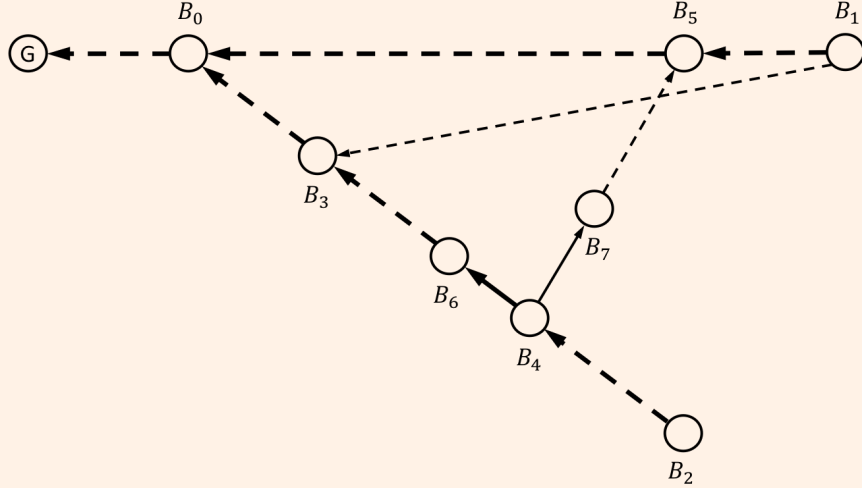


Figure 3: The case  $\text{ep}(B_0) = i$ . Solid and dashed lines represent parent-child links and ancestor-descendant links, respectively. Bold and regular lines represent  $\xrightarrow{b}$  and  $\rightarrow$  relations, respectively.

in assumption A5, the number of distinct committees which have issued at least one block in  $\mathcal{C}(B_0, B_1)$  is at most

$$2(N_i - K_i) < \frac{2}{3}N_i < K_i. \quad (5)$$

It then follows from Assumption A4 that  $|\mathcal{C}(B_0, B_1)| < K_i$ , which is equivalent to  $|\mathcal{C}(B_0, B_1)| \leq K_i - 1$ . It completes the proof of Lemma 1.  $\square$

**Lemma 2.** *If  $B_1 \xrightarrow{b} B_0$ ,  $\text{ep}(B_1) = i$  and  $B_1$  satisfies the condition (2) with respect to  $B_0$ , for any block  $B_2$  such that  $\text{ep}(B_2) = i$ ,  $B_2 \xrightarrow{b} B_0$  and  $\mathcal{C}(B_0, B_2) \cap \mathcal{C}(B_0, B_1) = \emptyset$ , we have  $\text{lv}(B_2) < \text{lv}(B_1)$ .*

*Proof.* Since  $\text{ep}(B_0) \leq \text{eq}(B_1) = i$  by Proposition 2, in the following we consider two cases, namely  $\text{ep}(B_0) = i$  or  $\text{ep}(B_0) < i$ .

First, consider the case  $\text{ep}(B_0) = i$ . It means that  $\mathcal{S}(B_0, B_1) \neq \emptyset$  since  $B_0 \in \mathcal{S}(B_0, B_1)$ . We start from  $B_2$ , traverse through best parent links till  $B_0$ , and stop as soon as a block in  $\mathcal{S}(B_0, B_1)$  is encountered. Let  $B_3$  denote the block we stop at, i.e.,

$$B_3 = \arg \max_{B \in (\mathcal{C}(B_0, B_2) \cup \{B_0\}) \cap \mathcal{S}(B_0, B_1)} \text{lv}(B). \quad (6)$$

We show that no block in  $\mathcal{C}(B_3, B_2)$  is issued by any honest committee from set  $\mathcal{W}(B_1)$ . It is proved by contradiction. Assume there are blocks in  $\mathcal{C}(B_3, B_2)$  issued by honest committees from  $\mathcal{W}(B_1)$ . Among those, let  $B_4$  denote the one with the smallest height. As shown in Fig. 3, let  $B_5$  denote the block in set  $\mathcal{T}(B_1)$  which comes from the same committee as  $B_4$ . Since  $B_4$  and  $B_5$  come from the same honest committee, by Assumption A1, either  $B_4$  includes  $B_5$  or  $B_5$  includes  $B_4$ . Since  $B_2$  includes  $B_3$  and  $\text{ep}(B_2) = \text{ep}(B_3) = i$ , we have  $\text{ep}(B_4) = i$  by Corollary 1. Similarly, we have  $\text{ep}(B_5) = \text{ep}(B_1) = i$ . Therefore, by Corollary 1, either  $B_4 \rightarrow B_5$  or  $B_5 \rightarrow B_4$  holds. However, by the definition of  $B_3$  in (6), which is the first block included by  $B_1$  when traversing from  $B_2$  through best parent links, it is impossible that  $B_5 \rightarrow B_4$ . Thus, we have  $B_4 \rightarrow B_5$ . Let  $B_6$  and  $B_7$  be parents of  $B_4$  such that  $B_4 \xrightarrow{b} B_6$  and  $B_7 \rightarrow B_5$ , respectively. Since  $\text{ep}(B_2) = \text{ep}(B_3) = i$ , all blocks in  $\mathcal{C}(B_3, B_6)$  are in epoch  $i$  by Corollary 1. By the definition of  $B_4$ , no block in  $\mathcal{C}(B_3, B_6)$  is issued by any honest committee from  $\mathcal{W}(B_1)$ . In addition, the cardinality of  $\mathcal{W}(B_1)$  is  $K_i$  since  $B_1$  satisfies the condition (2), which implies that  $\text{lv}(B_1) > K_i$ . Therefore, by Lemma 1, we have  $|\mathcal{C}(B_3, B_6)| \leq K_i - 1$ , which leads to

$$\text{lv}(B_6) \leq \text{lv}(B_3) + (K_i - 1). \quad (7)$$

Now the following chain of inequalities hold

$$\text{lv}(B_7) \stackrel{(a)}{\geq} \text{lv}(B_5) \stackrel{(b)}{\geq} \text{lv}(B_1) - (K_i - 1) \stackrel{(c)}{>} \text{lv}(B_3) + (K_i - 1) \stackrel{(d)}{\geq} \text{lv}(B_6), \quad (8)$$

where (a) is by Proposition 3, (b) is due to  $B_5 \in \mathcal{T}(B_1)$ , (c) is by the fact that  $B_3 \in \mathcal{S}(B_0, B_1)$  and  $B_1$  satisfies the condition (2) with respect to  $B_0$ , and (d) is by (7). It contradicts with the fact that  $\text{lv}(B_6) \geq \text{lv}(B_7)$  since  $B_6$  is the best parent of  $B_4$  and  $\text{ep}(B_6) = \text{ep}(B_7) = i$ . It completes the proof that no block in  $\mathcal{C}(B_3, B_2)$  is issued by any honest committee from  $\mathcal{W}(B_1)$ . In addition,  $B_2 \xrightarrow{b} B_3$  and all blocks in  $\mathcal{C}(B_3, B_2)$  are in epoch  $i$ , by Lemma 1 we have  $|\mathcal{C}(B_3, B_2)| \leq K_i - 1$ , which leads to

$$\text{lv}(B_2) \leq \text{lv}(B_3) + (K_i - 1). \quad (9)$$

It follows that

$$\text{lv}(B_1) \stackrel{(a)}{>} \text{lv}(B_3) + 2(K_i - 1) \stackrel{(b)}{\geq} \text{lv}(B_2) + (K_i - 1) \geq \text{lv}(B_2), \quad (10)$$

where (a) is by the fact that  $B_3 \in \mathcal{S}(B_0, B_1)$  and  $B_1$  satisfies the condition (2) with respect to  $B_0$ , and (b) is by (9). It completes the proof that  $\text{lv}(B_2) < \text{lv}(B_1)$  if  $\text{ep}(B_0) = i$ .

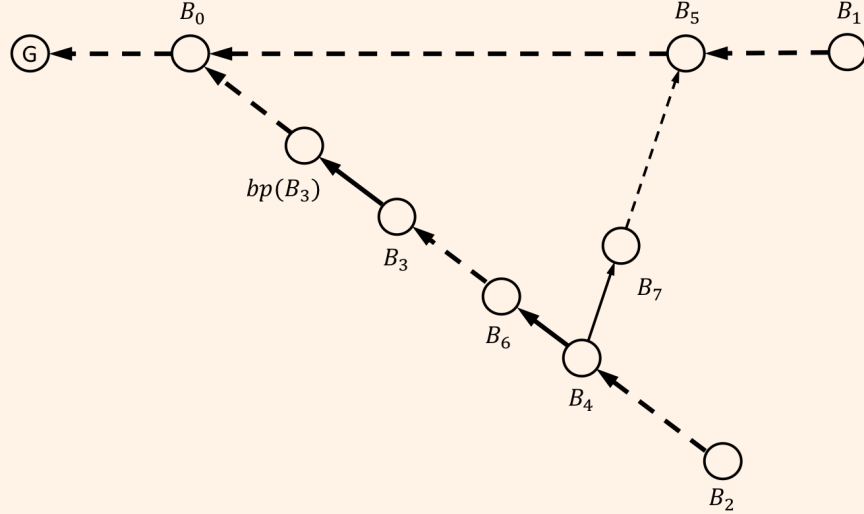


Figure 4: The case  $\text{ep}(B_0) < i$ . Solid and dashed lines represent parent-child links and ancestor-descendant links, respectively. Bold and regular lines represent  $\xrightarrow{b}$  and  $\rightarrow$  relations, respectively.

Next, we consider the case  $\text{ep}(B_0) < i$ . If  $\mathcal{S}(B_0, B_1) \neq \emptyset$ , we can follow the same arguments as in the previous proof to show that  $\text{lv}(B_2) < \text{lv}(B_1)$ . Now we assume  $\mathcal{S}(B_0, B_1) = \emptyset$ . Since  $\text{ep}(B_2) = i > \text{ep}(B_0)$ , by Proposition 1, there exists a block  $B_3 \in \mathcal{C}(B_0, B_2)$  such that  $\text{ep}(B_3) = i$  and  $\text{ep}(\text{bp}(B_3)) = i - 1$ , i.e.,  $\text{lv}(B_3) = 1$ . Similarly as in the previous case, we show that no block in  $\mathcal{C}(\text{bp}(B_3), B_2)$  is issued by any honest committee from set  $W(B_1)$ . It is also proved by contradiction. Assume there are blocks in  $\mathcal{C}(\text{bp}(B_3), B_2)$  issued by honest committees from  $W(B_1)$ . Among those, let  $B_4$  denote the one with the smallest height. As shown in Fig. 4, let  $B_5$  denote the block in set  $\mathcal{T}(B_1)$  which comes from the same committee as  $B_4$ . Since  $B_4$  and  $B_5$  come from the same honest committee, by Assumption A1, either  $B_4$  includes  $B_5$  or  $B_5$  includes  $B_4$ . Since  $B_2$  includes  $B_3$  and  $\text{ep}(B_2) = \text{ep}(B_3)$ , we have  $\text{ep}(B_4) = i$  by Corollary 1. Also we have  $\text{ep}(B_5) = \text{ep}(B_1) = i$ . Therefore, by Corollary 1, either  $B_4 \rightarrow B_5$  or  $B_5 \rightarrow B_4$  holds. However, it is impossible that  $B_5 \rightarrow B_4$  since it is assumed that  $\mathcal{S}(B_0, B_1) = \emptyset$ . Thus, we have  $B_4 \rightarrow B_5$ . Let  $B_6$  and  $B_7$  be parents of  $B_4$  such that  $B_4 \xrightarrow{b} B_6$  and  $B_7 \rightarrow B_5$ , respectively. If  $B_4 = B_3$ , we have

$$\text{ep}(B_6) = \text{ep}(\text{bp}(B_3)) = i - 1 < i = \text{ep}(B_5) \leq \text{ep}(B_7), \quad (11)$$

where the last inequality is due to Proposition 2. It contradicts with the

fact that  $B_6$  is the best parent of  $B_4$ . If  $B_4 \neq B_3$ , by the definition of  $B_4$ , no block in  $\mathcal{C}(\text{bp}(B_3), B_6)$  is issued by any honest committee from  $\mathcal{W}(B_1)$ . And all blocks in  $\mathcal{C}(\text{bp}(B_3), B_6)$  are in epoch  $i$ . Therefore, by Lemma 1, we have  $|\mathcal{C}(\text{bp}(B_3), B_6)| \leq K_i - 1$ , which leads to

$$\text{lv}(B_6) \leq K_i - 1, \quad (12)$$

since  $\text{lv}(B_3) = 1$ . In the following, we derive a similar chain of inequalities as (8):

$$\text{lv}(B_7) \stackrel{(a)}{\geq} \text{lv}(B_5) \stackrel{(b)}{\geq} \text{lv}(B_1) - (K_i - 1) \stackrel{(c)}{>} K_i - 1 \stackrel{(d)}{\geq} \text{lv}(B_6), \quad (13)$$

where (a) is by Proposition 3, (b) is due to  $B_5 \in \mathcal{T}(B_1)$ , (c) is by the fact that  $B_1$  satisfies the condition (2) which implies  $\text{lv}(B_1) > 2(K_i - 1)$ , and (d) is from (12). It contradicts with the fact that  $\text{lv}(B_6) \geq \text{lv}(B_7)$  since  $B_6$  is the best parent of  $B_4$  and  $\text{ep}(B_6) = \text{ep}(B_7) = i$ . It completes the proof that no block in  $\mathcal{C}(\text{bp}(B_3), B_2)$  is issued by any honest committee from  $\mathcal{W}(B_1)$ .

In addition,  $B_2 \xrightarrow{b} \text{bp}(B_3)$  and all blocks in  $\mathcal{C}(\text{bp}(B_3), B_2)$  are in epoch  $i$ , by Lemma 1 we have  $|\mathcal{C}(\text{bp}(B_3), B_2)| \leq K_i - 1$ , which leads to

$$\text{lv}(B_2) \leq K_i - 1, \quad (14)$$

since  $\text{lv}(B_3) = 1$ . It follows that

$$\text{lv}(B_1) \stackrel{(a)}{>} 2(K_i - 1) \stackrel{(b)}{\geq} \text{lv}(B_2) + (K_i - 1) \geq \text{lv}(B_2), \quad (15)$$

where (a) is by the fact that  $B_1$  satisfies the condition (2) which implies  $\text{lv}(B_1) > 2(K_i - 1)$ , and (b) is by (14). It completes the proof that  $\text{lv}(B_2) < \text{lv}(B_1)$  if  $\text{ep}(B_0) < i$ .

By combining the two cases above, we finish the proof of Lemma 2.  $\square$

**Lemma 3.** *Given  $i \in \mathbb{N}$ , assume  $\text{lsb}(B)$  is a stable block of graph  $G^B$  for any block  $B$  with  $\text{ep}(B) < i$ . If  $B_1 \xrightarrow{b} B_0$ ,  $\text{ep}(B_1) = i$ ,  $\text{h}(B_0) \in \mathcal{I}_i$  and  $B_1$  satisfies the condition (2) with respect to  $B_0$ , for any block  $B_2$  such that  $B_2 \xrightarrow{b} B_0$  and  $\mathcal{C}(B_0, B_2) \cap \mathcal{C}(B_0, B_1) = \emptyset$ , we have  $\text{ep}(B_2) \leq \text{ep}(B_1)$ .*

*Proof.* According to the procedure of determining the last stable block in D10, we have  $B_2 \xrightarrow{b} \text{lsb}(B_2)$ . Since  $B_2 \xrightarrow{b} B_0$ , either  $B_0 \xrightarrow{b} \text{lsb}(B_2)$  or  $\text{lsb}(B_2) \xrightarrow{b} B_0$  holds. We show that  $B_0 \xrightarrow{b} \text{lsb}(B_2)$ . It is proved by contradiction. Suppose  $\text{lsb}(B_2) \xrightarrow{b} B_0$  and  $\text{lsb}(B_2) \neq B_0$ , which means that the



last stable block of  $B_2$  has advanced past  $B_0$ . Thus, there exists some block  $B_3 \in \mathcal{C}(B_0, B_2)$  such that  $B_3$  satisfies the condition (2) with respect to  $B_0$ , i.e.,

$$\text{lv}(B_3) > \max_{B \in \mathcal{S}(B_0, B_3)} \text{lv}(B) + 2(K_j - 1), \quad (16)$$

where  $j = \text{ep}(B_3) \leq \text{ep}(B_2) = i$  by Proposition 2. And the last stable block of  $B_3$  has advanced past  $B_0$ , i.e.,  $\text{lsb}(B_3) \in \mathcal{C}(B_0, B_2)$ . Consider the following two cases.

- 1)  $j < i$ : Let  $\mathbf{G}^* = \mathbf{G}^{B_3} \cup \mathbf{G}^{B_1}$ . Since  $\text{ep}(B_3) < \text{ep}(B_1)$ ,  $B_1$  is the tip block of the main chain of  $\mathbf{G}^*$ . By the assumption in the statement of Lemma 3,  $\text{lsb}(B_3)$  is a stable block of graph  $\mathbf{G}^{B_3}$ . Due to  $\mathbf{G}^{B_3} \subseteq \mathbf{G}^*$ ,  $\text{lsb}(B_3)$  is on the main chain of  $\mathbf{G}^*$ , i.e.,  $B_1 \xrightarrow{b} \text{lsb}(B_3)$ . It contradicts with the fact that  $\text{lsb}(B_3) \in \mathcal{C}(B_0, B_2)$  and  $\mathcal{C}(B_0, B_2) \cap \mathcal{C}(B_0, B_1) = \emptyset$ .
- 2)  $j = i$ : Since both  $B_1$  and  $B_3$  satisfy the condition (2) with respect to  $B_0$ , it follows by Lemma 2 that both  $\text{lv}(B_3) < \text{lv}(B_1)$  and  $\text{lv}(B_1) < \text{lv}(B_3)$  hold, which is a contradiction.

Now we have shown that  $B_0 \xrightarrow{b} \text{lsb}(B_2)$ . In addition, we have  $\text{lsb}(B_2) \xrightarrow{b} \text{lsb}(\text{bp}(B_2))$  by Proposition 1. Thus,  $B_0 \xrightarrow{b} \text{lsb}(\text{bp}(B_2))$  holds. It follows that  $\text{h}(\text{lsb}(\text{bp}(B_2))) \leq \text{h}(B_0)$ . Since  $\text{h}(B_0) \in \mathcal{I}_i$ , there exists  $k \leq i$  such that  $\text{h}(\text{lsb}(\text{bp}(B_2))) \in \mathcal{I}_k$ , which leads to  $\text{ep}(B_2) = k \leq i = \text{ep}(B_1)$ . It completes the proof of Lemma 3.  $\square$

Now we can prove the following main result of this section.

**Theorem 1.** *For any block  $B_1$  in graph  $\mathbf{G}$ , the last stable block of  $B_1$ , i.e.,  $\text{lsb}(B_1)$  is a stable block of graph  $\mathbf{G}^{B_1}$ .*

*Proof.* We prove by induction. It is trivial for the case that  $B_1$  is the genesis block. For the case  $\text{ep}(B_1) = i$ , we assume that for any block  $B$  such that  $\text{ep}(B) < i$  or  $B = \text{bp}(B_1)$ ,  $\text{lsb}(B)$  is a stable block of  $\mathbf{G}^B$ . We will prove that  $\text{lsb}(B_1)$  is a stable block of graph  $\mathbf{G}^{B_1}$ .

We first show that for any block  $B_0$  such that  $B_0$  is a stable block of  $\mathbf{G}^{B_1}$ ,  $\text{h}(B_0) \in \mathcal{I}_i$ , and  $B_1$  satisfies the condition (2) with respect to  $B_0$ , then  $B_0$ 's child in  $\mathcal{C}(B_0, B_1)$ , denoted by  $B_0^*$ , is also a stable block of  $\mathbf{G}^{B_1}$ . It is equivalent to show that  $B_0^*$  is on the main chain of any graph  $\mathbf{G}^*$  such that  $\mathbf{G}^{B_1} \subseteq \mathbf{G}^*$ . We prove by contradiction. Assume there exists a graph  $\mathbf{G}^*$  such that  $\mathbf{G}^{B_1} \subseteq \mathbf{G}^*$  and the main chain of  $\mathbf{G}^*$  does not contain  $B_0^*$ . As depicted in Fig. 5, let  $B_2$  denote the tip block of the main chain of  $\mathbf{G}^*$ . Since  $B_0$

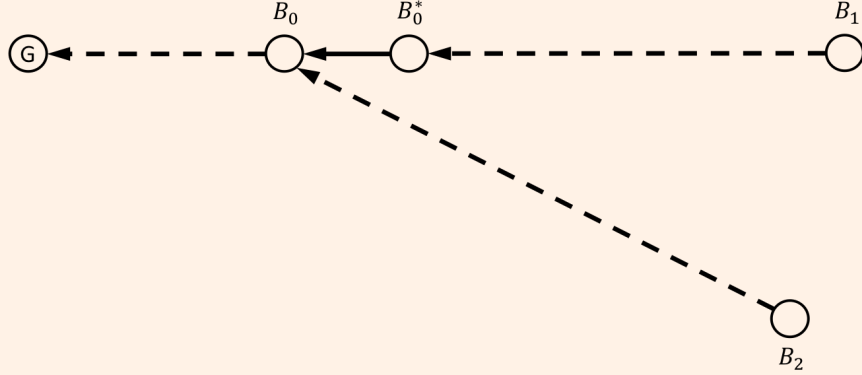


Figure 5: The case where  $B_0^*$  is not a stable block of  $\mathbf{G}^{B_1}$ . Solid and dashed lines represent parent-child links and ancestor-descendant links, respectively.

is a stable block of  $\mathbf{G}^{B_1}$  and  $\mathbf{G}^{B_1} \subseteq \mathbf{G}^*$ , the main chain of  $\mathbf{G}^*$  must contain  $B_0$ , i.e.,  $B_2 \xrightarrow{b} B_0$ . Now we have  $\mathcal{C}(B_0, B_2) \cap \mathcal{C}(B_0, B_1) = \emptyset$ . It follows that  $\text{ep}(B_2) \leq \text{ep}(B_1)$  by Lemma 3. Furthermore, if  $\text{ep}(B_2) = \text{ep}(B_1) = i$ , we have  $\text{lv}(B_2) < \text{lv}(B_1)$  by Lemma 2. Therefore, either  $\text{ep}(B_2) < \text{ep}(B_1)$  or  $\text{lv}(B_2) < \text{lv}(B_1)$  when  $\text{ep}(B_2) = \text{ep}(B_1)$  holds, which implies that  $B_1$  is better than  $B_2$  under block comparison rule  $\mathcal{R}$ . It contradicts with the fact that  $B_2$  is the tip block of the main chain of  $\mathbf{G}^*$  which contains both  $B_1$  and  $B_2$ .

We start with  $B_0 = \text{lsb}(\text{bp}(B_1))$ . Since  $\text{ep}(B_1) = i$ , we have  $\text{h}(B_0) \in \mathcal{I}_i$ . In addition,  $B_0$  is a stable block of  $\mathbf{G}^{\text{bp}(B_1)}$  by our assumption. And since  $\mathbf{G}^{\text{bp}(B_1)} \subseteq \mathbf{G}^{B_1}$ ,  $B_0$  is also a stable block of  $\mathbf{G}^{B_1}$ . Thus, by the result we have proved above,  $B_0$ 's child in  $\mathcal{C}(B_0, B_1)$ , denoted by  $B_0^*$ , is a stable block of  $\mathbf{G}^{B_1}$ . We set  $B_0$  to be  $B_0^*$ , and repeat this process until  $\text{h}(B_0) \notin \mathcal{I}_i$  or  $B_1$  does not satisfy the condition (2) with respect to  $B_0$ . The block we stop at, i.e., the last stable block of  $B_1$  is a stable block of  $\mathbf{G}^{B_1}$ . It completes the proof of Theorem 1.  $\square$

### 4.3 Safety

Recall that the local graph node  $i$  observes at time  $t$  is denoted by  $\mathbf{G}_i(t)$ . To determine the order of two blocks at time  $t$ , node  $i$  will first find the stable main chain of  $\mathbf{G}_i(t)$ , i.e.,  $\text{SC}(\mathbf{G}_i(t))$ , and then find out the order of these two blocks by rule  $\mathcal{O}$  in Section 2 given both of them have main chain indices (defined in D12). Therefore, in order to show the safety property of our consensus algorithm, it suffices to prove that the stable main chains

different nodes observe at different time are consistent, which is stated in the following Theorem 2.

**Theorem 2.** *For any  $i, j \in \mathbb{N}$  and  $t_i, t_j \geq 0$ , we have either  $\text{SC}(\mathbf{G}_i(t_i)) \subseteq \text{SC}(\mathbf{G}_j(t_j))$  or  $\text{SC}(\mathbf{G}_j(t_j)) \subseteq \text{SC}(\mathbf{G}_i(t_i))$ .*

*Proof.* Recall that  $\text{SB}(\mathbf{G}_i(t))$  denotes the tip block of the stable main chain node  $i$  observes at time  $t$ . We first show that  $\text{SB}(\mathbf{G}_i(t))$  is a stable block of graph  $\mathbf{G}_i(t)$ . In fact, by the definition of stable main chain in D11,  $\text{SB}(\mathbf{G}_i(t))$  can be represented as

$$\text{SB}(\mathbf{G}_i(t)) = \arg \max_{B \in \mathbf{G}_i(t)} h(\text{lsb}(B)). \quad (17)$$

For any  $B \in \mathbf{G}_i(t)$ , let  $\mathbf{G}_i^B(t)$  denote the induced graph which consists of all blocks included by  $B$ . By Theorem 1,  $\text{lsb}(B)$  is a stable block of  $\mathbf{G}_i^B(t)$ . For any graph  $\mathbf{G}^*$  such that  $\mathbf{G}_i(t) \subseteq \mathbf{G}^*$ , we have  $\mathbf{G}_i^B(t) \subseteq \mathbf{G}_i(t) \subseteq \mathbf{G}^*$ . It follows that  $\text{lsb}(B)$  is on the main chain of  $\mathbf{G}^*$ . Thus,  $\text{lsb}(B)$  is a stable block of  $\mathbf{G}_i(t)$ . Therefore, according to the definition in (17),  $\text{SB}(\mathbf{G}_i(t))$  is a stable block of  $\mathbf{G}_i(t)$ .

In order to prove that either  $\text{SC}(\mathbf{G}_i(t_i)) \subseteq \text{SC}(\mathbf{G}_j(t_j))$  or  $\text{SC}(\mathbf{G}_j(t_j)) \subseteq \text{SC}(\mathbf{G}_i(t_i))$  holds, it is equivalent to show that  $\text{SB}(\mathbf{G}_i(t_i)) \xrightarrow{b} \text{SB}(\mathbf{G}_j(t_j))$  or  $\text{SB}(\mathbf{G}_j(t_j)) \xrightarrow{b} \text{SB}(\mathbf{G}_i(t_i))$ . In fact, by Assumption A6, there exists some time  $t_j^*$  such that  $\mathbf{G}_i(t_i) \subseteq \mathbf{G}_j(t_j^*)$ . Let  $T = \max\{t_j, t_j^*\}$ . We have both  $\mathbf{G}_i(t_i) \subseteq \mathbf{G}_j(T)$  and  $\mathbf{G}_j(t_j) \subseteq \mathbf{G}_j(T)$ . Since  $\text{SB}(\mathbf{G}_i(t_i))$  is a stable block of  $\mathbf{G}_i(t_i)$ , it follows that  $\text{SB}(\mathbf{G}_i(t_i))$  is on the main chain of  $\mathbf{G}_j(T)$ . Similarly,  $\text{SB}(\mathbf{G}_j(t_j))$  is on the main chain of  $\mathbf{G}_j(T)$ . Therefore, due to the uniqueness of the main chain, we have either  $\text{SB}(\mathbf{G}_i(t_i)) \xrightarrow{b} \text{SB}(\mathbf{G}_j(t_j))$  or  $\text{SB}(\mathbf{G}_j(t_j)) \xrightarrow{b} \text{SB}(\mathbf{G}_i(t_i))$ . It completes the proof of Theorem 2.  $\square$

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